Differential Geometry I: Curves and surfaces

A. Standard exercises:

8.1 We consider the following functions: $f : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$f(x, y) = x^2y + 2x^2 - 2xy - 4x + y \quad \text{and} \quad g(x, y, z) = 2xy - 3yz.$$

(a) For which values of $c \in \mathbb{R}$ is the level curve $f^{-1}(c)$ a submanifold of $\mathbb{R}^2$ (and of what dimension)?

(b) For which values of $c \in \mathbb{R}$ is the level surface $g^{-1}(c)$ a submanifold of $\mathbb{R}^3$ (and of what dimension)?

Solution. We will make use of the regular value theorem: if $f : U \subset \mathbb{R}^n \to \mathbb{R}$ is of class $\mathcal{C}^k$ and if $df_p \neq 0$ at every point p such that $f(p) = c$, then $f^{-1}(c)$ is a submanifold of class $\mathcal{C}^k$ (one can replace the differential by the gradient). Of course, for those values c for which df_p vanishes at some point $p \inf^{-1}(c)$, the above theorem does not allow us to conclude anything about $f^{-1}(c)$; checking whether it is a submanifold in that case will have to be done explicitly.

(a) We compute the gradient of f:

$$\vec{\nabla} f(x, y) = \begin{pmatrix} 2xy + 4x - 2y - 4 \\ x^2 - 2x + 1 \end{pmatrix} = \begin{pmatrix} 2(x-1)(y+2) \\ (x-1)^2 \end{pmatrix}.$$

This gradient is zero if and only if $x = 1$, and in that case we compute $f(x, y) = f(1, y) = -2$. Consequently, $f^{-1}(c)$ is a submanifold of $\mathbb{R}^2$ if $c \neq -2$ as an application of the regular value theorem. For $c = -2$, we have to explicitly check the structure of $f^{-1}(-2)$. Note that, from the above calculation, we have that $\{x = 1\} \subset f^{-1}(-2)$. Moreover, it is easy to check that, for $y = -2$, we also have $f(x, -2) = -2$ for all x. Therefore, $f^{-1}(-2)$ contains the union of intersecting prthogonal lines $\{x = 1\} \cup \{y = -2\}$, hence it is not a submanifold (since in no neighborhood $\mathcal{U}$ of the point $(1, 2)$ is $\mathcal{U} \cap f^{-1}(-2)$ homeomorphic to an interval; if we remove the point $(1, 2)$ from $f^{-1}(-2) \cap \mathcal{U}$, where $\mathcal{U}$ is an arbitrarily small open neighborhood of $(1, -2)$ in $\mathbb{R}^2$, then the resulting set has at least four connected components, while an interval minus a point has two connected components).

(b) We compute the gradient of g:

$$\vec{\nabla} g(x, y, z) = \begin{pmatrix} 2y \\ 2x - 3z \\ -3y \end{pmatrix}.$$

This gradient vanishes exactly on the set of multiples of $(3, 0, 2)$, i.e. on the line $D = \mathbb{R} \cdot (3, 0, 2)$, and we note that on this line we also have $g(x, y, z) = 0$. Consequently, if $c \neq 0$ then $g^{-1}(c)$ is a submanifold of $\mathbb{R}^3$ (since then $\vec{\nabla} g \neq 0$).

However, $g^{-1}(0)$ is not a submanifold since

$$(x, y, z) \in g^{-1}(0) \Leftrightarrow g(x, y, z) = y(2x - 3z) = 0 \Leftrightarrow y = 0 \text{ or } 2x - 3z = 0.$$

This set is the union of two transverse planes and is therefore not a submanifold. On this line g is identically zero. In particular, $g^{-1}(0)$ is not a submanifold.

Remark. In these arguments one could have replaced the gradient by the differential.

8.2 **(a)** Let $p = (x_0, y_0, z_0) \in \mathbb{R}^3$ be a regular point of the surface S defined by the equation $f(x, y, z) = 0$. Prove that the tangent vector plane T_pS is the plane orthogonal to the gradient $\vec{\nabla} f(p)$.

(b) The affine tangent plane to a surface S at a regular point p is the set of points of $\mathbb{R}^3$ such that the vector $\overrightarrow{pq} \in T_pS$. Show that the affine tangent plane is given by

$$A_pS = \{q \in \mathbb{R}^3 \mid \langle q - p, \vec{\nabla} f(p)\rangle = 0\}.$$

(c) By applying the previous result, obtain the formula giving the first-order approximation of a differentiable function of two variables $z = \varphi(x, y)$ in the neighborhood of a point (x_0, y_0) (Taylor expansion of order 1).

Solution. (a) Recall that T_pS is the set of tangent vectors at the point p to all curves drawn on the surface, i.e. $v \in T_pS$ if and only if there exists a curve $\gamma : (-\varepsilon, \varepsilon) \to S$ such that $\gamma(0) = p$ and $\dot{\gamma}(0) = v$. If we write $\gamma(t) = (x(t), y(t), z(t))$, then we have

$$f(x(t), y(t), z(t)) \equiv 0 \Rightarrow 0 = \frac{d}{dt} f(x(t), y(t), z(t)) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}.$$

Therefore, if $v = \dot{\gamma}(0) = (v_1, v_2, v_3)$, then

$$\langle \vec{\nabla} f(p), v\rangle = v_1 \frac{\partial f}{\partial x}(p) + v_2 \frac{\partial f}{\partial y}(p) + v_3 \frac{\partial f}{\partial z}(p) = 0.$$

This shows that $T_pS \subset (\vec{\nabla} f(p))^\perp = \ker(df(p))$. In fact, this is an equality, because on the one hand T_pS is a vector subspace of dimension 2 (by the implicit function theorem), and on the other hand $\ker(df(p))$ is also a vector subspace of dimension 2 since we assumed that p is a regular point.

(b) Evident from the definitions.

(c) The graph of φ satisfies the equation $f(x, y, z) = 0$, where $f(x, y, z) = z - \varphi(x, y)$. Thus, since

$$\vec{\nabla} f(p) = \left(-\frac{\partial \varphi}{\partial x}, -\frac{\partial \varphi}{\partial y}, 1\right),$$

the equation of the affine tangent plane at a point $p = (x_0, y_0, z_0) = (x_0, y_0, \phi(x_0, y_0)) \in \{f = 0\}$ is therefore

$$-(x - x_0)\frac{\partial \varphi}{\partial x} - (y - y_0)\frac{\partial \varphi}{\partial y} + (z - z_0) = 0,$$

which can be reexpressed as (since $z_0 = \varphi(x_0, y_0)$):

$$z = \varphi(x_0, y_0) + \frac{\partial \varphi}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial \varphi}{\partial y}(x_0, y_0)(y - y_0),$$

The Taylor expansion for ϕ around (x_0, y_0) yields that the graph of ϕ satisfies the relation

$$z = \varphi(x_0, y_0) + \frac{\partial \varphi}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial \varphi}{\partial y}(x_0, y_0)(y - y_0) + o(|x - x_0| + |y - y_0|).$$

Therefore, we verify that the graph of the affine plane is the graph of the first order Taylor approxi-amtion of the function at the corresponding point.

8.3 Show that the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

is a regular surface (i.e. a submanifold of dimension 2) and compute its affine tangent plane at a point $p = (x_0, y_0, z_0)$.

Solution. The equation defining the ellipsoid is $S = \Big\{ f(x, y, z) = 0 \Big\}$, where

$$f(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1.$$

We have $\nabla f(x, y, z) = \left(\frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2}\right)$, therefore f has only one critical point at $(0, 0, 0)$; this point is not on the ellipsoid, which (by the regular value theorem) is therefore a regular surface.

The affine tangent plane at the point p is

$$A_p S = \{q \in \mathbb{R}^3 : \langle \nabla f(p), q - p \rangle = 0\}.$$

Setting $p = (x_0, y_0, z_0)$ and $q = (x, y, z)$, we obtain

$$\langle \nabla f(p), q-p \rangle = \left\langle \left(\frac{2x_0}{a^2}, \frac{2y_0}{b^2}, \frac{2z_0}{c^2} \right), (x - x_0, y - y_0, z - z_0) \right\rangle = \frac{2x_0(x - x_0)}{a^2} + \frac{2y_0(y - y_0)}{b^2} + \frac{2z_0(z - z_0)}{c^2}.$$

We can divide by 2, and we obtain the equation

$$A_p S := \{(x, y, z) \in \mathbb{R}^3 \mid \frac{x_0}{a^2}(x - x_0) + \frac{y_0}{b^2}(y - y_0) + \frac{z_0}{c^2}(z - z_0) = 0\}.$$

8.4 Two differentiable submanifolds $\mathcal{M}_1$ and $\mathcal{M}_2$ of $\mathbb{R}^n$ are said to intersect transversally at a point p if $p \in \mathcal{M}_1 \cap \mathcal{M}_2$ and at this point the tangent spaces satisfy $T_p\mathcal{M}_1 + T_p\mathcal{M}_2 = \mathbb{R}^n$.

- **(a)** Give an example of a surface and a regular curve in $\mathbb{R}^3$ that intersect at a single point, but in a non-transversal way.
- **(b)** Show that if S is a surface and C a curve in $\mathbb{R}^3$ (both regular), which intersect transversally at $0 \in \mathbb{R}^3$, then one can construct a system of local coordinates (u, v, t) in a neighborhood of 0 such that (u, v) are local parameters of the surface S and t a local parameter of the curve C.
- **(c)** In the same situation as in (b), prove that 0 is an isolated point of the intersection $S \cap C$ (i.e. there exists an open set $\mathcal{V} \subset \mathbb{R}^3$ such that $\mathcal{V} \cap S \cap C = \{0\}$).

Remark: In the above, saying that a curve or a surface is regular means that it is a submanifold of class $\mathcal{C}^k$, with $k \geqslant 1$.

Solution. (a) We can take for example a sphere and a line tangent to this sphere. Say $S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ and $L = \{(x, y, z) \in \mathbb{R}^3 \mid z = 0 \text{ and } y = 1\}$.

(b) The surface S and the curve C are assumed to be regular (they are submanifolds). They pass through the point $p = 0$, and we can therefore locally parametrize these manifolds by $\mathcal{C}^k$ maps (where k is the regularity class of the submanifolds):

$$\gamma : I \to C \subset \mathbb{R}^3 \quad \text{and} \quad \psi : \Omega \to S \subset \mathbb{R}^3,$$

where $I \subset \mathbb{R}$ is an interval containing 0 and we have $\gamma(0) = 0$ and $\dot{\gamma}(0) \neq 0$, and $\Omega \subset \mathbb{R}^2$ is an open set containing $(0, 0)$ and we have $\psi(0, 0) = 0$ and $d\psi_{0,0}$ injective (equivalently, the partial derivative vectors $b_1 = \frac{\partial\psi}{\partial u_1}$ and $b_2 = \frac{\partial\psi}{\partial u_2}$ at $(0, 0)$ are linearly independent).

Now consider the map

$$\Phi : \Omega \times I \to \mathbb{R}^3, \quad \Phi(u_1, u_2, t) = \psi(u_1, u_2) + \gamma(t).$$

We observe that $\Omega \times I$ is an open subset of $\mathbb{R}^3$ that contains the origin, and its differential at $(0, 0, 0)$ is nonzero because the vectors

$$\frac{\partial\Phi}{\partial u_1}(0, 0, 0) = b_1(0, 0), \quad \frac{\partial\Phi}{\partial u_2}(0, 0, 0) = b_2(0, 0), \quad \frac{\partial\Phi}{\partial t}(0, 0, 0) = \dot{\gamma}(0)$$

are linearly independent by the transversality hypothesis. The inverse function theorem then tells us that there exists an open neighborhood $\mathcal{U} \subset \Omega \times I$ of $(0, 0, 0)$ such that $\Phi : \mathcal{U} \to \Phi(\mathcal{U}) \subset \mathbb{R}^3$ is a diffeomorphism. By restricting the parameter domains Ω and I if necessary, we may assume that $\mathcal{U} = \Omega \times I$. We have thus constructed the desired local coordinate system (u_1, u_2, t) on the open set $\mathcal{V} = \Phi(\mathcal{U}) = \Phi(\Omega \times I)$.

(c) Let $p \in S \cap C$. With the previous notation, we clearly have for the open neighborhood $\mathcal{V}$ of p

$$S \cap \mathcal{V} = \Phi(\{(u_1, u_2, t) \in U \mid t = 0\}) \quad \text{and} \quad C \cap \mathcal{V} = \Phi(\{(u_1, u_2, t) \in U \mid u_1 = u_2 = 0\}).$$

Thus

$$S \cap C \cap \mathcal{V} = \Phi(\{(u_1, u_2, t) \in U \mid t = u_1 = u_2 = 0\}) = \{(0, 0, 0)\}.$$

8.5 The window of Viviani is the intersection curve of a sphere with a right circular cylinder that passes through the center of the sphere and whose diameter is equal to the radius of the sphere. If the radius of the sphere is 2, we can therefore assume (up to applying an isometry) that Viviani's window is defined by the equations:

$$x^2 + y^2 + z^2 = 4 \quad \text{and} \quad (x - 1)^2 + y^2 = 1.$$

We denote this set by V.

(a) Show by a geometric argument that there exists a point $q \in V$ such that the complement $V \setminus \{q\}$ is a differentiable submanifold of $\mathbb{R}^3$. What are the coordinates of q (we accept a heuristic argument)?

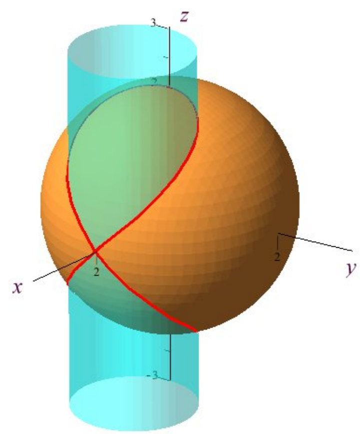

Figure 1: A depiction of Viviani's window. Credit: Wikipedia.

(b) Prove rigorously from the equations of V that $V \setminus \{q\} \subset \mathbb{R}^3$ is a differentiable submanifold.

(c) Find a regular parametrization of this curve.

Solution. (a) Our first "geometric" (aka intuitive) explanation will be based on the fact that the intersection of two surfaces is a submanifod around those points where the surfaces intersect transversally (try to prove this fact rigorously using the regular value theorem). At every point of the cylinder, the tangent plane is a vertical plane. For the sphere, the only vertical tangent planes are the tangent planes at the points of the equator $\{z = 0\}$. There is only one point of Viviani's window lying on the equator, namely $q = (1, 0, 0)$. We conclude that the two surfaces intersect transversally at every point of $V \setminus \{q\}$, and thus $V \setminus \{q\}$ is a differentiable submanifold of dimension 1 of $\mathbb{R}^3$.

(b) Let $f(x, y, z) = x^2 + y^2 + z^2 - 1$ and $g(x, y, z) = \left(x - \frac{1}{2}\right)^2 + y^2 - \frac{1}{4}$, then $p \in V$ if and only if $f(p) = g(p) = 0$. Compute the gradients (or the differentials if preferred):

$$\nabla f = 2(x, y, z), \quad \nabla g = 2\left(x - \frac{1}{2}, y, 0\right);$$

we see that ∇f and ∇g are linearly dependent if and only if $z = y = 0$. Adding the conditions $f = g = 0$, we must have $x = +1$. This shows that at every point of $V \setminus \{(1, 0, 0)\}$, the Jacobian matrix

$$\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \end{pmatrix}$$

has constant rank $= 2$, and therefore $V \setminus \{(1, 0, 0)\}$ is a differentiable submanifold of $\mathbb{R}^3$ (of codimension 2, hence of dimension $1 = 3 - 2$).

(c) To parametrize Viviani's window, we must reflect a bit on its geometry. We observe that it is a closed curve whose projection onto the Oxy plane travels twice around the circle of radius $1/2$ and center $(1/2, 0)$. It therefore seems reasonable to try the parametrization

$$\gamma(t) = \left(\frac{1}{2}(\cos(2t) + 1), \frac{1}{2}\sin(2t), z(t)\right), \quad t \in [0, 2\pi],$$

where $z(t)$ is a function to be determined. We note that for any function $z(t)$, the curve γ lies on the cylinder; in particular, $g(\gamma(t)) = 0$ (we used $\cos(2t)$ and $\sin(2t)$ because the projection of γ travels twice around the circle of radius $1/2$ centered at $(1/2, 0)$). To determine $z(t)$, we must use that $\gamma(t)$ lies on the unit sphere, that is, $f(\gamma(t)) = 0$, hence $z^2(t) = 1 - x^2(t) - y^2(t)$. We compute

$$z^2(t) = 1 - x^2(t) - y^2(t) = 1 - \frac{1}{4}(\cos(2t)+1)^2 - \frac{1}{4}\sin^2(2t) = 1 - \frac{1}{4}\big(\cos^2(2t) + 2\cos(2t) + 1\big) - \frac{1}{4}\sin^2(2t) = \frac{1}{2}(1 - \cos(2$$

Hence we can parametrize the curve V by

$$\gamma(t) = \left(\frac{1}{2}(\cos(2t)+1), \frac{1}{2}\sin(2t), \sin(t)\right), \quad t \in [0, 2\pi],$$

we note that $z(t) \geqslant 0$ when $0 \leqslant t \leqslant \pi$ and $z(t) \leqslant 0$ when $\pi \leqslant t \leqslant 2\pi$.

Finally, the velocity vector of γ is $\dot\gamma(t) = (-\sin(2t), \cos(2t), \cos(t))$ and the speed is $\|\dot\gamma\| = \sqrt{1+\cos^2(t)} \geqslant 1$ for all t, hence the parametrization is regular, including at the double point $q = \gamma(0) = \gamma(\pi)$.

B. Bonus exercise:

8.6 We have seen in Exercise 7.4 that $O(n)$ and $SL_n(\mathbb{R})$ are submanifolds of $M_n(\mathbb{R})$.

(a) Describe the tangent space $T_{\mathrm{I}}SL_n(\mathbb{R})$ to the submanifold $SL_n(\mathbb{R}) \subset M_n(\mathbb{R})$ at the point I (the identity matrix).

(b) Describe the tangent space $T_{\mathrm{I}}O(n)$ to the submanifold $O(n) \subset M_n(\mathbb{R})$ at the point I.

Solution. (a) Referring to Exercise 7.4(c) (recall that $SL_n(\mathbb{R})$ was found to be a submanifold of $\mathcal{M}_n(\mathbb{R})$ by being the level set $f(A) = 1$ of $f(A) = \det(A)$) and applying the theorem describing the tangent space $T_p\mathcal{M}$ of the submanifold $\mathcal{M} = \{f = \text{const}\}$ at p as the vector subspace $\ker(df_p)$, we infer that that the tangent space $T_I SL_n(\mathbb{R})$ is the kernel of the linear form

$$H \mapsto d(\det)_I(H) = \mathrm{Trace}(H)$$

(see Ex. 7.1 for the differential of the determinant). Therefore, the tangent space at I to $SL_n(\mathbb{R})$ is the set of matrices of trace zero (also denoted $\mathfrak{sl}_n$ in the context of classical or Lie groups).

(b) Recall that near $O(n)$, the map

$$\Phi : A \mapsto AA^T$$

has constant rank (see the solution of Ex. 7.4(d)). Its differential at $A = I$ is given by

$$d\Phi_I(H) = H + H^T.$$

The group $O(n)$ is defined by the equation $\Phi(A) = I$ (i.e. as the level set $\Phi^{-1}(I)$). Thus, its tangent space at the identity is therefore

$$T_I O(n) = \ker d_I\Phi = \{H \in M_n(\mathbb{R}) \mid H + H^T = 0\},$$

which is the set of antisymmetric matrices (also denoted $\mathfrak{o}(n)$ in the context of Lie groups).